Combinatorial Networks
May 20th-21st,Wednesday

- **Definition.** For graph G,let
  $\alpha(G)$ = the largest size of the independent set (IS) in G,
  $\omega(G)$ = the largest size of the clique in G,
  $\chi(G)$ = the chromatic number of G.

- **Fact1.** (1) $\chi(G) \geq \omega(G)$; (2) $|V(G)| \leq \alpha(G)\chi(G)$.

- **Definition.** A graph is *perfect* if for any induced subgraph H of G,we have $\chi(H) = \omega(H)$.

- **Fact2.** Bipartite graph G is perfect.

- **Lemma1.** If G is bipartite,then $\chi(G^c) = \omega(G^c)$

  Proof. Let $G = (A, B), |A| = n, |B| = m$, M and M^* are the matching and the maximum matching of G, VC and VC^* are the vertex-cover and the minimum vertex-cover of G,then we have

  - $n + m - |M| \geq \chi(G^c)$;
  - $\omega(G^c) \geq n + m - |VC|$.

  So
  $$n + m - |M^*| \geq \chi(G^c) \geq \omega(G^c) \geq n + m - |VC^*| \overset{\text{König's Theorem}}{=} n + m - |M^*|.$$
  Then we get
  $$\chi(G^c) = \omega(G^c) = n + m - |M^*| n + m - |VC^*|.$$
  ∎

- **Remark:** König's Theorem: For bipartite graph G, $\max|M| = \min|VC|$.

- **Thm1(Weak Perfect Graph Theorem:)** G is perfect iff G^c is perfect.

  Proof. We use the Theorem:

  G is perfect iff for any induced subgraph H of G, $|V(H)| \leq \alpha(H)\omega(H)$.(We will prove it later.)

  G is perfect $\Leftrightarrow$ for any induced subgraph H of G,$|V(H)| \leq \alpha(H)\omega(H)$ $\Leftrightarrow$ for any induced subgraph H^c of G^c,$|V(H^c)| \leq \omega(H^c)\alpha(H^c)$ $\Leftrightarrow$ G^c is perfect. ∎

- **Thm2:** G is perfect iff $\underbrace{\text{for any induced subgraph } H \text{ of } G,\ |V(H)| \leq \alpha(H)\omega(H)}_{*}$.

  Proof.

  $\Rightarrow$: G is perfect, then for any induced subgraph $H, \chi(H) = \omega(H)$. By Fact1, $|V(G)| \leq \alpha(G)\chi(G) = \alpha(H)\omega(H)$.

  $\Leftarrow$: By induction on $|V(G)|$.

  Suppose every graph with less than n vertices satisfying $(*)$ is perfect.

Suppose G satisfies $(*)$ but is not perfect,then $\chi(G) > \omega(G)$ because $(*)$ is monotone, that is **if $G_1 \subseteq G_2$ and G_2 has $(*)$, then G_1 has $(*)$.**

We will show $n = |V(G)| \geq \alpha(G)\omega(G) + 1$.

- **Claim1:** Suppose U is an IS in G, then $\chi(G \setminus U) = \omega(G \setminus U) = \omega(G)$.

  Proof. It's easy to get $\chi(G \setminus U) = \omega(G \setminus U)$ by induction.

  Clearly, $\omega(G \setminus U) \leq \omega(G)$.

  Suppose not "=", then

$$\chi(G \setminus U) = \omega(G \setminus U) \leq \omega(G) - 1 \leq (\chi(G) - 1) - 1.$$

  And we have $\chi(G) \leq \chi(G \setminus U) + 1$.

  So $\chi(G) - 1 \leq \chi(G) - 2$. A contradiction!

  That means $\omega(G \setminus U) = \omega(G)$.

  From now on, we denote $\omega(G) = \omega, \alpha(G) = \alpha$.

- **Claim2:** Let U be an IS and K be a clique with size ω in G, if $K \cap U = \emptyset$, then in any $\omega-$coloring of $G \setminus U$, K intersects in every color class by exactly one vertex; if $K \cap U \neq \emptyset$ (ie. $|K \cap U| = 1$), then K intersects all but one of the color classes of any $\omega-$coloring of $G \setminus U$ by exactly one vertex.

  Let $U_0 = \{v_1, v_2, \ldots, v_\alpha\}$ be an IS of G with size $\alpha = \alpha(G)$.

  Let $U_{(i-1)\omega+1}, U_{(i-1)\omega+2}, \cdots, U_{(i-1)\omega+\omega}$ be the color-classes of a $\omega-$coloring in $G \setminus \{v_i\}$, $i = 1, 2 \cdots, \alpha$. All together, we have $\alpha\omega + 1$ IS.

  For $0 \leq i \leq \alpha\omega$, let K_i be a clique in $G \setminus U_i$ of size ω.

- **Claim3:** $\forall i \neq j, |K_i \cap U_j| = 1, (i, j = 0, 1, 2, \cdots, \alpha\omega)$.

  Proof.

  (1) for $i = 0, j = (p-1)\omega + q, p = 1, 2, \cdots, \alpha, q = 1, 2, \cdots, \omega$,

  since $K_0 \cap U_0 = \emptyset$, so $K_0 \cap \{v_p\} = \emptyset$. By Claim2, $|K_0 \cap U_j| = 1$.

  (2) for $j = 0, i = (p-1)\omega + q, p = 1, 2, \cdots, \alpha, q = 1, 2, \cdots, \omega$,

  since $v_p \in K_i$ (By Claim2) and $v_p \in K_0$, so $K_i \cap U_0 \neq \emptyset$. Since $|K_i \cap U_0|$ is not more than 1, then $|K_i \cap U_0| = 1$.

  (3) for $i = (p_1 - 1)\omega + q_1, j = (p_2 - 1)\omega + q_2, p_1, p_2 = 1, 2, \cdots, \alpha, q_1, q_2 = 1, 2, \cdots, \omega$, and $i \neq j$,

  when $p_1 = p_2$, since $K_i \subseteq G \setminus U_i$, then $K_i \cap U_i = \emptyset$. So $|K_i \cap U_{(p_2-1)\omega+1}|, |K_i \cap U_{(p_2-1)\omega+2}|, \cdots, |K_i \cap U_{(p_2-1)\omega+\omega}|$ all are one, that is $|K_i \cap U_j| = 1$.

when $p_2 \neq p_1$, if $K_i \cap U_j = \emptyset$, then $v_{p_2} \in K_i$. But $v_{p_1} \in K_i$, a contradiction! So $|K_i \cap U_j| = 1$.

Let $A = (a_{ij})$ is a $n \times (\alpha\omega + 1)$ matrix with

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \in U_j, \\ 0, & \text{otherwise.} \end{cases}$$

Let $B = (b_{ij})$ is a $(\alpha\omega + 1) \times n$ matrix with

$$b_{ij} = \begin{cases} 1, & \text{if } v_j \in K_i, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$BA = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}_{(\alpha\omega+1)(\alpha\omega+1)} = J - I$$

So $n \geq \text{rank}(A) \geq \text{rank}(BA) = \alpha\omega + 1$. This contradiction completes the proof. ∎

- **Remark:**

$$Det(aI_n + bJ_n) = \begin{vmatrix} a+b & b & \cdots & b \\ b & a+b & \cdots & b \\ \vdots & \vdots & & \vdots \\ b & b & \cdots & a+b \end{vmatrix} = (a + nb)a^{n-1}$$

So $Det(BA) = \alpha\omega(-1)^{\alpha\omega} \neq 0$. It means that $\text{rank}(BA) = \alpha\omega + 1$.